Universität Konstanz
Fachbereich Mathematik und Statistik
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Exercises for the course "Linear Algebra I"

Sheet 12

Hand in your solutions on Thursday, 30. Januar 2020, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 12.1 *(4 points)*

Let A be an arbitrary non-empty set and let $R \subseteq A \times A$ be a relation on A.

(a) Let R be an equivalence relation. For any $x \in A$ we call the set $[x]_R := \{y \in A \colon xRy\}$ the equivalence class of x with respect to R. Show that for all $x, y \in A$ either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$. Moreover, show that $A = \sqcup_{x \in A}[x]_R$.

(b) Conversely, prove the following statement: Let I be a set and let $\{A_i : i \in I\}$ be a family of non-empty subsets of A such that

(i) $A_i \cap A_j = \emptyset$ for all $i, j \in I$ such that $i \neq j$ and

(ii) $A = \sqcup_{i \in I} A_i$.

Then the following defines an equivalence relation on A:

$$xRy :\Leftrightarrow \exists i \in I(\{x, y\} \subseteq A_i)$$

Exercise 12.2 *(4 points)*

Let K be an arbitrary field, $n \geq 2$ arbitrary and $M = \mathrm{Mat}_{n\times n}(K)$. Decide in (a) and (b) whether the given relation R on M is reflexive, symmetric and transitive, and whether it is an equivalence relation.

(a) $(A, B) \in R$ if and only if A and B are row equivalent.

(b) $(A, B) \in R$ if and only if there is some matrix $C \in M$ such that $A = BC$.

($\star$) Describe the relation R from part (a) in the form $A = BC$ for some appropriate matrix C.

(c) Check the following argument:

Let M be a set and R a symmetric and transitive relation on M. We claim that then R is also reflexive. So let $x \in M$. Choose some $y \in M$ such that xRy. By symmetry we get yRx. From xRy and yRx follows via transitivity that xRx. Hence, R is reflexive.

Where is the mistake? Find a concrete counterexample.

Exercise 12.3 *(4 points)*

Consider the following linearly independent vectors in $\mathbb{R}^{3\times 1}$:

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Determine the dual basis of $\{u_1, u_2, u_3\}$, i.e. the linear mappings $f_i \in (\mathbb{R}^3)^*$ such that $f_i(u_j) = \delta_{ij}$ $(i = 1, 2, 3)$.

Exercise 12.4 *(4 points)*

Let K be a field and $n \in \mathbb{N}$. For any $A = (a_{ij})_{1\leq i,j\leq n} \in Mat_{n\times n}(K)$, the sum $\mathrm{tr}(A) := \sum_{i=1}^{n} a_{ii}$ is called the *trace of* A. Show:

(a) $\mathrm{tr}\colon Mat_{n\times n}(K) \to K$ is a K-linear map.

(b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in Mat_{n\times n}(K)$.

(c) The following are equivalent:

(i) For any $A \in Mat_{n\times n}(K)$ there is some $a \in K$ such that $\mathrm{tr}(A - aI_n) = 0$.

(ii) $char(K)$ does not divide n.


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